

# Elliptic stochastic quantisation

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Stochastic quantization is based on a relation between stationary solutions to SDEs and Gibbs measures.

Consider the SDE

$$\partial_t \varphi(t) + \partial V(\varphi(t)) = \sqrt{2} \xi(t)$$

where  $\varphi(t) \in \mathbb{R}^n$ ,  $V: \mathbb{R}^n \rightarrow \mathbb{R}$  is a positive smooth function growing fast enough at infinity and  $\xi(t)$  is a white noise in time  $t$ .

It can be easily proven using the Fokker-Planck equation that the invariant solution to the previous SDE is such that

$$\varphi(t) \sim \frac{\exp(-V(\eta)) d\eta}{\mathcal{Z}}.$$

In the '80, a similar phenomenon was *discovered* by physicists for invariant solutions to elliptic SPDEs with additive noise.

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Purely probabilistic statement: the invariant solution of the following *elliptic* SPDE

$$(-\Delta_x + m^2)\varphi(x) + \partial V(\varphi(x)) = \xi(x), \quad x \in \mathbb{R}^2$$

where  $x \in \mathbb{R}^2$ ,  $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^n$  and  $V: \mathbb{R}^n \rightarrow \mathbb{R}$  is a non-negative smooth function bounded from below, should satisfy

$$\varphi(x) \sim \frac{1}{\mathcal{Z}} \exp \left[ -4\pi \left( \frac{m^2 |\eta|^2}{2} + V(\eta) \right) \right] d\eta.$$

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The statement is a particular case of a more general phenomenon, which connects to the topic of stochastic quantisation of Euclidean QFT.

The SPDE

$$(-\Delta_x - \Delta_z + m^2)\varphi(x, z) + \partial V(\varphi(x, z)) = \xi(x, z), \quad x \in \mathbb{R}^2, z \in \mathbb{R}^d$$

where  $x \in \mathbb{R}^2$ ,  $\varphi: \mathbb{R}^{d+2} \rightarrow \mathbb{R}$  and  $V$  is a non-negative smooth function bounded from below, should satisfy

$$\varphi(x, \cdot) \sim \frac{1}{\mathcal{Z}} \exp \left[ -4\pi \left( \int_{\mathbb{R}^d} \frac{1}{2} |\nabla \phi(z)|^2 + \frac{m^2}{2} |\phi(z)|^2 + V(\phi(z)) dz \right) \right] \mathcal{D} \phi.$$

where on the r.h.s.  $\phi: \mathbb{R}^d \rightarrow \mathbb{R}$ .

This is called *elliptic stochastic quantisation*.

Initially perturbative computations via Feynman diagrams, in the context of some field theoretic approximation of the random field Ising model.

Then Parisi and Sourlas gave a non-perturbative “proof” via a **supersymmetric approach** (susy).

Dimensional reduction: a supersymmetric euclidean QFT of bosons and fermions in  $d+2$  dimensions is equivalent to a bosonic theory in  $d$  dimensions.

In this original formulation has been subject of study of multiple papers in the physics and mathematics literature.

The form involving elliptic SPDEs treated here was proposed as an alternative to the more famous parabolic stochastic quantization and as such it was described mainly from an heuristic point view.

A partial rigorous discussion of this topic was given in a paper by A. Klein, L. J. Landau, and J. F. Perez in the '80.

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In the first part of the talk we propose a proof of the previous conjecture, for a special class of potentials  $V$ , which closely follows the original idea of Parisi and Sourlas stressing the role of supersymmetry.

In the second part we give an overview of the extensions obtained for more general potentials.

Join work with S. Albeverio and F. C. De Vecchi:

- S. Albeverio, F. C. De Vecchi and M. G. Elliptic stochastic quantization. *Annals of Probability* 48(4):1693–1741, 2020.
- —. The elliptic stochastic quantization of some two dimensional Euclidean QFTs. [arXiv:1906.11187](https://arxiv.org/abs/1906.11187), 2019 (to appear in *Ann. I.H.P. PS*).
- F. C. De Vecchi and M. G. A Note on Supersymmetry and Stochastic Differential Equations. 2019.

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Let  $V$  be a smooth bounded function with  $-|\partial^2 V| + m^2 > \epsilon$ .

Consider the SPDE

$$(-\Delta + m^2)\varphi(x) + f(x) \partial V(\varphi(x)) = \xi(x), \quad x \in \mathbb{R}^2,$$

where  $f: \mathbb{R}^2 \rightarrow \mathbb{R}_{\geq 0}$  is a smooth, radial, *cutoff* function depending decaying exponentially at infinity with  $f(0) = 1$ . Then, for any  $h: \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$\mathbb{E} [h(\varphi(0)) Y_f(\varphi)] = \frac{\mathcal{Z}_f}{\mathcal{Z}} \int_{\mathbb{R}^n} h(y) \exp \left[ -4\pi \left( \frac{m^2 |\eta|^2}{2} + V(\eta) \right) \right] d\eta$$

where

$$Y_f(\varphi) = \exp \left[ \int_{\mathbb{R}^2} f'(x) V(\varphi(x)) dx \right], \quad \mathcal{Z}_f = \mathbb{E}[Y_f(\varphi)].$$

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The first step in the proof is to rewrite the expectation of the solution to the elliptic equation as an integral with respect to a Gaussian measure.

For this reason we introduce the map  $U: \mathcal{W} \rightarrow \mathcal{W}$  (where  $\mathcal{W}$  is a suitable Banach space) of the form

$$U(\phi) = \mathcal{I}(f \partial V(\phi))$$

with  $\mathcal{I} = (-\Delta + m^2)^{-1}$ . Let

$$T(\phi) = \phi + U(\phi)$$

and rewrite the equation as

$$T(\varphi) = \mathcal{I}\xi.$$



This means that if  $\nu$  is the law of  $\varphi$  we have

$$T_{\star}\nu = \mu$$

where  $\mu$  is the (Gaussian) law of  $\mathcal{T}\xi$ . Note that  $\mathcal{T}\xi$  is a massive Gaussian free field in  $\mathbb{R}^2$ .

When  $V$  is a convex and bounded and  $f$  is a smooth non-negative function decaying exponentially to 0 at infinity we have:

- $U(\phi) \in \mathcal{T}(L^2(\mathbb{R}^2))$
- $U$  is differentiable and  $\nabla U(\phi) = \mathcal{T}(f \partial^2 V(\phi))$
- $\nabla U(\phi)$  is Hilbert-Schmidt on  $\mathcal{T}(L^2(\mathbb{R}^2))$
- $T$  is invertible

We can apply the theory of general shifts in abstract Wiener spaces and prove that  $\nu$  is absolutely continuous with respect to  $\mu$  and

$$d\nu(\phi) = \det_2(I + \nabla U(\phi)) \exp \left[ -\delta(U) - \frac{1}{2} \|U\|_{\mathcal{H}(L^2(\mathbb{R}^2))}^2 \right] d\mu(\phi).$$

[see e.g. A. S. Üstünel, Analysis on Wiener Space and Applications. [ArXiv:1003.1649](https://arxiv.org/abs/1003.1649), 2010.]

This implies that

$$\begin{aligned} \mathcal{L}_f \mathcal{I}_h &:= \mathbb{E} [h(\varphi(\mathbf{o})) Y_f(\varphi)] = \int_{\mathcal{W}} h(\phi(\mathbf{o})) Y_f(\phi) d\nu(\phi) \\ &= \int_{\mathcal{W}} h(\phi(\mathbf{o})) Y_f(\phi) \det_2(I + \nabla U(\phi)) \exp \left[ -\delta(U) - \frac{1}{2} \|U\|_{\mathcal{H}(L^2(\mathbb{R}^2))}^2 \right] d\mu(\phi). \end{aligned}$$

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Formally the superspace  $\mathfrak{S}$  is the set of points  $(x, \theta, \bar{\theta})$  where  $x \in \mathbb{R}^2$  and  $\theta, \bar{\theta}$  are two additional anticommuting coordinates. It is denoted  $\mathbb{R}^{2|2}$ .

Two bosonic and two fermionic coordinates. It is a non-commutative space.

A more concrete construction is to understand  $\mathfrak{S}$  via the algebra of smooth functions  $F: \mathfrak{S} \rightarrow \mathbb{C}$  on it.

Let  $\mathfrak{G}(\theta, \bar{\theta})$  be the Grassmann algebra generated by symbols  $\theta, \bar{\theta}$ , with the relations

$$\theta\bar{\theta} = -\bar{\theta}\theta, \quad \theta^2 = \bar{\theta}^2 = 0,$$

Then  $\mathfrak{G}(\theta, \bar{\theta}) = \text{span}(1, \theta, \bar{\theta}, \theta\bar{\theta})$  and a smooth function  $F: \mathbb{R}^2 \rightarrow \mathfrak{G}(\theta, \bar{\theta})$  is just a quadruplet

$$(f_{\emptyset}, f_{\theta}, f_{\bar{\theta}}, f_{\theta\bar{\theta}}) \in (C^{\infty}(\mathbb{R}^2))^4$$

with the identification

$$F(x) = f_{\emptyset}(x) + f_{\theta}(x)\theta + f_{\bar{\theta}}(x)\bar{\theta} + f_{\theta\bar{\theta}}(x)\theta\bar{\theta}.$$

The function  $F$  can be considered as a function  $F: \mathfrak{G} \rightarrow \mathbb{R}$  on the superspace  $\mathfrak{G}$  by formally writing

$$F(x, \theta, \bar{\theta}) := F(x) = f_{\varnothing}(x) + f_{\theta}(x)\theta + f_{\bar{\theta}}(x)\bar{\theta} + f_{\theta\bar{\theta}}(x)\theta\bar{\theta}.$$

We can introduce a linear functional defined as

$$F \mapsto \int F(x, \theta, \bar{\theta}) dx d\theta d\bar{\theta} := - \int_{\mathbb{R}^2} f_{\theta\bar{\theta}}(x) dx,$$

induced by the standard Berezin integral  $\alpha \in \mathfrak{G} \mapsto \int \alpha d\theta d\bar{\theta} \in \mathbb{C}$  on  $\mathfrak{G}$  satisfying

$$\int 1 d\theta d\bar{\theta} = \int \theta d\theta d\bar{\theta} = \int \bar{\theta} d\theta d\bar{\theta} = 0, \quad \int \theta\bar{\theta} d\theta d\bar{\theta} = -1.$$

Given  $r \in C^1(\mathbb{R}; \mathbb{R})$  we define the composition  $r \circ F: \mathfrak{G} \rightarrow \mathbb{R}$  by

$$r(F(x, \theta, \bar{\theta})) := r(f_\emptyset(x)) + r'(f_\emptyset(x)) f_\theta(x) \theta + r'(f_\emptyset(x)) f_{\bar{\theta}}(x) \bar{\theta} + r'(f_\emptyset(x)) f_{\theta\bar{\theta}}(x) \theta \bar{\theta},$$

in accordance with the same expression one would get if  $r$  were a polynomial.

Moreover we can define similarly the space of Schwartz superfunctions  $\mathcal{S}(\mathfrak{G})$  and the Schwartz superdistributions  $T \in \mathcal{S}'(\mathfrak{G})$  where  $T$  can be written

$$T = T_\emptyset + T_\theta \theta + T_{\bar{\theta}} \bar{\theta} + T_{\theta\bar{\theta}} \theta \bar{\theta}$$

with  $T_\emptyset, T_\theta, T_{\bar{\theta}}, T_{\theta\bar{\theta}} \in \mathcal{S}'(\mathbb{R}^2)$  and duality pairing

$$T(f) = -T_\emptyset(f_{\theta\bar{\theta}}) + T_\theta(f_{\bar{\theta}}) - T_{\bar{\theta}}(f_\theta) - T_{\theta\bar{\theta}}(f_\emptyset), \quad f_\emptyset, f_\theta, f_{\bar{\theta}}, f_{\theta\bar{\theta}} \in \mathcal{S}(\mathbb{R}^2).$$

The superspace formalism allow to rewrite the integral  $\mathfrak{I}_h$  in a surprising form.

**Step 1.** We introduce an approximation  $\mathfrak{I}_h^\chi$  of  $\mathfrak{I}_h$  having the following expression

$$\mathfrak{L}_f^\chi \mathfrak{I}_h^\chi := \int h(\phi(o)) \underbrace{\det(I_{\mathfrak{H}} + \nabla U_\chi)}_{\text{Step 3}} \exp \left( -\delta \circ U_\chi - \underbrace{\frac{1}{2} \|U_\chi\|_{\mathcal{I}^\chi(L^2)}^2}_{\text{Step 2}} + 4 \int V(\phi(x)) f'(x) dx \right) d\mu^\chi(\phi)$$

where  $U_\chi = \varpi \mathcal{I}^\chi(\partial V(\phi))$ ,  $\mathcal{I}^\chi = (-\Delta + m^2)^{-1-\chi}$ ,  $\mu^\chi$  is the law of  $\mathcal{I}^\chi \xi$  and  $\varpi = \frac{1}{1+2\chi}$  is a constant.

**Step 2.** We introduce an auxiliary Gaussian field  $\eta$  distributed as the Gaussian noise  $\mathcal{I}^\chi \xi$  with law  $\mu^\chi$ , and independent with respect to  $\phi$ . We can then write

$$\exp \left[ -\frac{\varpi^2}{2} \|\mathcal{I}^\chi(f \partial V(\phi))\|_{\mathfrak{H}}^2 \right] = \int \exp \left[ -i \varpi \langle f \partial V(\phi), \mathcal{I}^\chi \eta \rangle \right] \mu^\chi(d\eta).$$

**Step 3.** We also introduce two free fermion fields  $\psi, \bar{\psi}$  realized as bounded operators on a suitable Hilbert space  $H_{\psi, \bar{\psi}}$  with a state  $\langle \cdot \rangle_{\psi, \bar{\psi}}$  for which

$$\langle \bar{\psi}(x)\bar{\psi}(x') \rangle_{\psi, \bar{\psi}} = \langle \psi(x)\psi(x') \rangle_{\psi, \bar{\psi}} = 0, \quad \langle \psi(x)\bar{\psi}(x') \rangle_{\psi, \bar{\psi}} = \omega \mathcal{G}_{I+2\chi}(|x-x'|^2),$$

where  $\mathcal{G}_{I+2\chi}(|x|^2)$  is the Green function of the operator  $(-\Delta + m^2)^{-I-2\chi}$ . From which we have

$$\det \left( I_{\mathcal{H}} + \omega f \partial^2 V(\phi) \mathcal{T}^\chi \right) = \left\langle \exp \left[ \int \psi(x) f(x) \partial^2 V(\phi(x)) \bar{\psi}(x) dx \right] \right\rangle_{\psi, \bar{\psi}}.$$

We take the generators  $\theta, \bar{\theta}$  to anticommute with the the fermion fields  $\psi, \bar{\psi}$ .

**Step 4.** We introduce the complex Gaussian field  $\omega := -\omega((m^2 - \Delta)\phi + i\mathcal{T}^\chi\eta)$  so that

$$\int f(x) \partial V(\phi(x)) \omega(x) dx = \left[ -\delta_\circ(\omega \mathcal{T}^\chi(\partial V(\phi))) - i\omega \langle f \partial V(\phi), \mathcal{T}^\chi \eta \rangle \right]$$

We put all our fields together in a single object defining the *superfield*  $\Phi$ :

$$\Phi(x, \theta, \bar{\theta}) := \phi(x) + \bar{\psi}(x)\theta + \psi(x)\bar{\theta} + \omega(x)\theta\bar{\theta},$$

A non-commutative random field, define on a non-commutative space.

$$\mathcal{Z}_f^X \mathcal{J}_h^X = \left\langle \left( - \int h(\Phi(X)) (\theta\bar{\theta}\delta_o(x)) dx d\theta d\bar{\theta} \right) \exp \left[ - \int V(\Phi(X)) \tilde{f}(|X|^2) dX \right] \right\rangle_X$$

where  $X = (x, \theta, \bar{\theta})$ ,  $dX = dx d\theta d\bar{\theta}$ ,  $|X|^2 = |x|^2 + 4\theta\bar{\theta}$ .

$$V(\Phi(X)) = V(\phi(x)) + \partial V(\phi(x)) (\bar{\psi}(x)\theta + \psi(x)\bar{\theta} + \omega(x)\theta\bar{\theta}) - \partial^2 V(\phi(x)) \bar{\psi}(x)\psi(x)\theta\bar{\theta}$$

$$\int V(\Phi(X)) \tilde{f}(|X|^2) dX = \int 4V(\phi(x)) f'(x) + f(x) \left( \underbrace{\partial^2 V(\phi(x)) \bar{\psi}(x)\psi(x)}_{\det} + \underbrace{\partial V(\phi(x)) \omega(x)}_{-\delta \circ U_X^{-\frac{1}{2}} \|U_X\|_{\mathcal{G}X(L^2)}^2} \right) dx$$



A supergroup: we introduce the linear transformations  $\tau(b, \bar{b}): \mathfrak{S} \rightarrow \mathfrak{S}$

$$\tau(b, \bar{b}) \begin{pmatrix} x \\ \theta \\ \bar{\theta} \end{pmatrix} = \begin{pmatrix} x + 2\bar{b}\theta\rho + 2b\bar{\theta}\rho \\ \theta - (x \cdot b)\rho \\ \bar{\theta} + (x \cdot \bar{b})\rho \end{pmatrix} \in \mathfrak{S}(\theta, \bar{\theta}, \rho)$$

for  $b, \bar{b} \in \mathbb{R}^2$  and where  $\rho$  is a new odd variable anticommuting with  $\theta, \bar{\theta}$  and we have

$$\tau(tb, t\bar{b})\tau(sb, s\bar{b}) = \tau((t+s)b, (t+s)\bar{b}).$$

We say that  $F$  is supersymmetric if it is invariant with respect to rotations in  $\mathbb{R}^2$  and transformation of the form  $\tau(b, \bar{b})$ , i.e. if

$$F(X) = F(\tau(b, \bar{b})X)$$

Supersymmetric distributions are defined by duality.

For supersymmetric functions and distribution the following fundamental theorem holds:

**Theorem.** *Let  $F \in \mathcal{S}(\mathfrak{S})$  be a smooth supersymmetric function then we have that there exists a function  $f: \mathbb{R}_+ \rightarrow \mathbb{R}$  such that*

$$F(x, \theta, \bar{\theta}) = f(|x|^2 + 4\theta\bar{\theta}).$$

*Suppose furthermore that  $T \in \mathcal{S}'(\mathfrak{S})$  is supersymmetric and such that  $T_\emptyset$  is a continuous function, then we have the reduction formula*

$$\int T(x, \theta, \bar{\theta}) \cdot F(x, \theta, \bar{\theta}) dx d\theta d\bar{\theta} = 4\pi T_\emptyset(o) F_\emptyset(o).$$

By direct verification computation we have that

$$\langle \Phi(x, \theta, \bar{\theta}) \Phi(x', \theta', \bar{\theta}') \rangle = C_{\Phi}(x - x', \theta - \theta', \bar{\theta} - \bar{\theta}')$$

where

$$C_{\Phi}(x, \theta, \bar{\theta}) := \mathcal{G}_{2+2\chi}(|x|^2) - \omega \mathcal{G}_{1+2\chi}(|x|^2) \theta \bar{\theta} = \mathcal{G}_{2+2\chi}(|x|^2 + 4\theta \bar{\theta}) \quad [!]$$

where  $\mathcal{G}_{2+2\chi}(|x|^2)$  is the Green function of the operator  $(-\Delta + m^2)^{-2-2\chi}$ .

**Striking fact:** The field  $\Phi(x, \theta, \bar{\theta})$  is supersymmetric “in law”.

For non-commutative r.v. the concept of law is not well defined, but we can show that expectations of supersymmetric functionals of  $\Phi(x, \theta, \bar{\theta})$  are supersymmetric.

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**Theorem.** Let  $V_1, V_2, \dots, V_n$  be smooth functions such that they and their derivatives grow at most polynomially, and let  $F_1, \dots, F_n \in \mathcal{S}$  be supersymmetric functions then

$$\left\langle \prod_{i=1}^n \left[ \int V_i(\Phi(x, \theta, \bar{\theta})) F_i(x, \theta, \bar{\theta}) dx d\theta d\bar{\theta} \right] \right\rangle_{\chi} = \left\langle \prod_{i=1}^n [4\pi f_{\phi}^i(o) V_i(\phi(o))] \right\rangle_{\chi}.$$

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Using the previous theorem, it is simple to prove, by that if  $V$  is bounded with bounded first and second derivatives we have

$$\begin{aligned}
 \mathcal{L}_f^\chi \tilde{\mathcal{I}}_h^\chi &= \left\langle \left( - \int h(\Phi(x, \theta, \bar{\theta})) (\theta \bar{\theta} \delta_o(x)) dx d\theta d\bar{\theta} \right) \times \right. \\
 &\quad \left. \times \exp \left[ - \int V(\Phi(x, \theta, \bar{\theta})) \tilde{f} (|x|^2 + 4 \theta \bar{\theta}) dx d\theta d\bar{\theta} \right] \right\rangle_\chi \\
 &= \left\langle \int h(\phi(o)) e^{-4\pi V(\phi(o))} \right\rangle_\chi \\
 &= \int_{\mathbb{R}^n} h(\eta) \exp \left[ -4\pi \left( \frac{(1+2\chi)}{2} m^{2(1+2\chi)} |\eta|^2 + V(\eta) \right) \right] d\eta / \mathcal{L}
 \end{aligned}$$

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There are two main problems in extending the previous results to more general potentials:

- when  $V$  is not convex the elliptic SPDE admits multiple solutions and so there is no clear relation between its solutions and the integral

$$\int h(\phi(o)) Y_f \Lambda_U d\mu,$$

- even if we focus only on this integral, we cannot extend the proof involving the supersymmetry since

$$\Lambda_{U^x} \not\rightarrow \Lambda_U$$

in  $L^p$  when  $V$  is a general smooth function.

The first step in the extension of dimensional reduction consists in proving some results for the integral

$$\int h(\phi(\mathbf{o})) Y_f \Lambda_U d\mu.$$

**Theorem 1. (Klein, Landau, Perez)** *Let  $V$  be an even polynomial convex at infinity then*

$$|\Lambda_U| \in L^p$$

and

$$\begin{aligned} \mathcal{L}_f^{-1} \int_{\mathcal{W}} h(\phi(\mathbf{o})) Y_f(\phi) \det_2(I + \nabla(U)(\phi)) \exp \left[ -\delta(U) - \frac{1}{2} \|U\|_{\mathcal{Y}(L^2(\mathbb{R}^2))}^2 \right] d\mu(\phi) \\ = \int_{\mathbb{R}^n} h(\eta) \exp \left[ -4\pi \left( \frac{1}{2} m^2 |\eta|^2 + V(\eta) \right) \right] d\eta / \mathcal{L} \end{aligned}$$

**Hypothesis  $V_\lambda$ .** We have the decomposition

$$V = V_B + \lambda V_U, \quad V_U(y) = \sum_{i=1}^n (y^i)^4, \quad y = (y^1, \dots, y^n) \in \mathbb{R}^n,$$

with  $\lambda > 0$  and  $V_B$  a bounded function with all bounded derivatives on  $\mathbb{R}^n$ .

**Theorem 2.** *If  $V$  satisfies hypothesis  $V_\lambda$  then  $\Lambda_U \in L^p$  and*

$$\int_{\mathcal{W}} h(\phi(o)) Y_f(\phi) \Lambda_U(\phi) d\mu(\phi) = \frac{\mathcal{L}_f}{\mathcal{L}} \int_{\mathbb{R}^n} h(\eta) \exp \left[ -4\pi \left( \frac{1}{2} m^2 |\eta|^2 + V(\eta) \right) \right] d\eta$$



The second step consists in extending the relation between the solutions to the elliptic SPDE and the integral  $\int h(\phi(o)) Y_f \Lambda_U d\mu$  when the potential is non-convex.

**Definition 3.** *The measure  $\nu$  of  $\varphi$  is a weak solution to the elliptic SPDE if*

$$T_\star \nu = \mu$$

where  $T(w) = w + \mathcal{T}(f \partial V(w))$ ,  $\mu$  is the law of  $\mathcal{T}\xi$  and  $\mathcal{T} = (-\Delta + m^2)^{-1}$ .

When  $V$  is non-convex, pathwise uniqueness is not clear, and there might be multiple weak solutions.

However, if  $V$  satisfies hypothesis  $V_\lambda$  it is possible to give a characterization of the generic weak solution.

**Theorem 4.** *The measure  $\nu$  is a weak solution to the elliptic SPDE if and only if*

$$\frac{d\nu}{d\mu} = A|\Lambda_U| = A|\det_2(I + \nabla U)| \exp\left(-\delta(U) - \frac{1}{2}\|U\|_{\mathcal{H}(L^2)}^2\right),$$

where  $A$  is any non-negative measurable function such that

$$\sum_{\phi' \in T^{-1}(\phi)} A(\phi') = 1$$

for  $\mu$ -almost  $\phi \in \mathcal{W}$ .

(via the general theory of Üstünel–Zakai)

**Lemma 5.** *If  $V$  satisfies hypothesis  $V_\lambda$  there exists at least a weak solution  $\nu_0$  such that*

$$\int_{\mathcal{W}} h(\phi(o)) Y_f(\phi) d\nu_0(\phi) = \frac{\mathcal{Z}_f}{\mathcal{Z}} \int_{\mathbb{R}^n} h(\eta) \exp\left(-4\pi\left(\frac{1}{2}m^2|\eta|^2 + V(\eta)\right)\right) d\eta$$

**Idea of the proof.** We identify the measure  $\nu_0$  with the positive functional

$$L(g) = \int g(\phi) d\nu_0(\phi)$$

defined on the Banach space  $C_b^0(\mathcal{W})$  of continuous bounded functions on  $\mathcal{W}$ .

The measure  $\nu_0$  satisfies the thesis of the theorem if and only if

$$L(\tilde{g} \circ T) = \int_{\mathcal{W}} \tilde{g}(\phi) d\nu_0(\phi)$$
$$L(h(\phi(0)) Y_f) = \frac{\mathcal{L}_f}{\mathcal{L}} \int_{\mathbb{R}^n} h(y) \exp\left(-4\pi\left(\frac{1}{2}m^2|\eta|^2 + V(\eta)\right)\right) d\eta.$$

It is possible to prove, using the previous theorem on

$$\int h(\phi(0)) Y_f \Lambda_U d\mu,$$

that the two previous conditions on  $L$  are compatible.

Thus the thesis follows exploiting the fact that  $\Lambda_U \in L^p$  and using the extension theorem for positive functionals on Riesz spaces.

The main idea of the third step consists in exploiting the properties of the elliptic SPDE.

We introduce the following hypothesis on  $V$ .

**Hypothesis QC.** The potential  $V: \mathbb{R}^n \rightarrow \mathbb{R}$  is a positive smooth function, such that it and its first and second partial derivatives grow at most exponentially at infinity and such that there exists a function  $H: \mathbb{R}^n \rightarrow \mathbb{R}$  with exponential growth at infinity such that we have

$$-\langle \hat{n}, \partial V(y + r\hat{n}) \rangle \leq H(y), \quad \hat{n} \in \mathbb{S}^{n-1}, y \in \mathbb{R}^n \text{ and } r \in \mathbb{R}_+,$$

with  $\mathbb{S}^{n-1}$  is the  $n-1$  dimensional sphere.

The following families of functions satisfy Hypothesis **QC**:

- Smooth convex functions,
- Smooth bounded functions,
- Smooth functions having the second derivative semidefinite positive outside a compact set,
- Any positive linear combinations of the previous functions.

Applying the maximum principle to the random equation

$$(-\Delta + m^2)\bar{\varphi} + f \partial V(\mathcal{J} \xi + \bar{\varphi}) = 0,$$

where  $\bar{\varphi} = \varphi - \mathcal{J} \xi$ , and using hypothesis **QC** on the potential  $V$  we obtain the a priori PDE bound:

$$\|\bar{\varphi}\|_{\infty} \lesssim \|H(\mathcal{J} \xi) f\|_{\infty}.$$

It is important to note that the previous inequality depends on  $V$  only through the function  $H$ . The idea is to approximate the potential  $V$  with a sequence of potentials  $(V_i)_i$ .

It is possible to prove the existence of a sequence of potentials  $V_i$  converging, uniformly on compact sets of  $\mathbb{R}^n$ , to  $V$  and satisfying hypothesis  $V_{\lambda}$  and hypothesis **QC** with the same function  $H$ .

These facts imply that:

- the sequence  $T_i$  converges to  $T$  uniformly on compact sets of  $\mathcal{W}$ ,
- any sequence of measures  $\nu_i$  such that  $T_{i,\star}(\nu_i) = \mu$  is tight,
- any weak limit  $\nu$  of a subsequence  $\nu_{i_n}$  is such that  $T_{\star}(\nu) = \mu$ .

We consider the weak limit  $\nu_0$  of a sequence of weak solutions  $\nu_{i_0}$ , satisfying the dimensional reduction theorem, then  $\nu_0$  will satisfy the dimensional reduction theorem.

**Theorem 6.** *Suppose that  $V$  satisfies the hypothesis **QC** and  $f$  satisfies the hypothesis **CO**, then there exists at least a weak solution  $\nu_0$  such that*

$$\mathbb{E}_{\nu_0} [h(\varphi(0)) Y_f(\varphi)] = \frac{\mathcal{L}_f}{\mathcal{L}} \int_{\mathbb{R}^n} h(\eta) \exp \left[ -4\pi \left( \frac{1}{2} m^2 |\eta|^2 + V(\eta) \right) \right] d\eta.$$



The main difficulty in this operation is to prove the convergence  $\mathcal{L}_f^{-1}Y_f(\varphi) \rightarrow \mathbf{I}$ .

Indeed this factor does not actually converge and what we can reliably expect is that

$$\lim_{f \rightarrow \mathbf{I}} \mathcal{L}_f^{-1} \mathbb{E}[Y_f(\varphi_f) \mid \sigma(\varphi_f(0))] \rightarrow \mathbf{I}.$$

**Theorem.** Suppose that  $\partial^2 V + m^2 > \epsilon$  let  $\varphi$  be the solution to the elliptic equation with  $f \equiv \mathbf{I}$ . Then for any  $x \in \mathbb{R}^2$  and any measurable and bounded function  $h$  defined on  $\mathbb{R}^n$  we have

$$\mathbb{E}[h(\varphi(x))] = \mathcal{L}^{-1} \int_{\mathbb{R}^n} h(\eta) \exp\left(-4\pi \left(\frac{\mathbf{I}}{2} m^2 |\eta|^2 + V(\eta)\right)\right) d\eta.$$

The idea of the prove is to consider the equation

$$(-\Delta + m^2)\varphi_t(x) + t \partial V_T(\varphi_t(x)) = \xi(x),$$

when  $V_T$  is a trigonometric polynomial and  $t \in \mathbb{R}$  such that

$$t \partial^2 V_T + m^2 > 0.$$

Under this hypothesis, and the condition that  $h$  is a polynomial, it is possible to prove that  $\varphi_t(x)$  is real analytic with respect to  $t$  and so  $\mathfrak{H}(t) = \mathbb{E}[h(\varphi_t(\mathbf{o}))]$  is real analytic with respect to  $t$ .

Then we are able to prove that

$$\partial_t^k \mathfrak{H}(\mathbf{o}) = \partial_t^k \left\{ \mathcal{L}_t^{-1} \int_{\mathbb{R}^n} h(\eta) \left[ -4\pi \left( \frac{1}{2} m^2 |\eta|^2 + t V_T(\eta) \right) \right] d\eta \right\} \Big|_{t=0}.$$

Since it is possible to approximate any potential satisfying hypothesis **QC** and such that  $\partial^2 V + m^2 > \epsilon$  with potentials  $V_T$  of the previous form the theorem is proved.

[ ]

We use the previous results to prove a version of dimensional reduction for elliptic SPDEs in dimension  $d+2$  for  $d > 0$ .

In particular we consider the equation

$$(-\Delta_x - \Delta_z + m^2)\varphi + \alpha_\epsilon^{*2} * (f(x')g(z')\partial V(\varphi)) = \alpha_\epsilon * (\xi(x, z))$$

where  $(x, z) \in \mathbb{R}^2 \times \mathbb{R}^d$ ,  $\alpha_\epsilon: \mathbb{R}^d \rightarrow \mathbb{R}$  is a mollifier,  $g$  is a smooth function with compact support on  $\mathbb{R}^d$  and  $\xi$  is a  $d+2$  dimensional white noise.

It is possible to prove that there exists at least a weak solution  $v_0$  to the previous equation such that

$$\exp \left[ \int_{\mathbb{R}^{d+2}} f'(x) g(z) \varphi(x, z) dx dz \right] dv_0 \Big|_{\varphi(0, \cdot)} \approx \exp \left[ -4\pi \left( K_\epsilon(\phi) + \int_{\mathbb{R}^d} g V(\phi) dz \right) \right] \mathcal{D} \phi$$

where  $\mathcal{D} \phi$  is a formal Lebesgue measure on the space of functions  $\phi: \mathbb{R}^d \rightarrow \mathbb{R}$  and

$$K_\epsilon(\phi) = \int_{\mathbb{R}^d} \left( |\mathcal{A}_\epsilon^{-1} \Delta_z \phi|^2 + m^2 \phi^2 \right) dz,$$

where  $\mathcal{A}_\epsilon(h) = a_\epsilon * h$ .

For  $d=1$  it is possible to take the limit  $a_\epsilon \rightarrow \delta_0$  since  $(-\Delta + m^2)^{-1} \xi$  is a continuous function.

For  $d \geq 2$  the equation becomes singular when  $\epsilon \rightarrow 0$ , and we need renormalization for defining the composition of  $\varphi$  with the function  $V$ .

In the case  $d=2$  and  $V = \exp(\alpha x)$  (for  $|\alpha| < 4\pi$ ) the renormalization leaves  $V$  convex.

Then we show that the solution to the singular elliptic SPDE

$$(-\Delta_x - \Delta_z + m^2)\varphi + \alpha :e^{\alpha\varphi} := \xi,$$

where  $: \exp(\alpha\varphi) := \exp\left(\alpha\varphi - \frac{\mathbb{E}[\alpha^2\varphi^2]}{2}\right)$ , has the law

$$\varphi(x, z) \sim \exp\left[-4\pi\left(K_o(\phi) + \int_{\mathbb{R}^d} : \exp(\alpha\phi(z)) : dz\right)\right] \mathcal{D}\phi$$

[ ]

- IR cut-off removal in the non-convex case;
- Invariance with respect to translations and uniqueness in non-convex case;
- Possible extensions of dimensional reduction to elliptic SPDEs with multiplicative noise;
- Probabilistic proof of dimensional reduction, explaining the role of supersymmetry.

[ ]

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